UDC 532.5

NOTE ON THE STABILITY OF THREE-DIMENSIONAL FLOWS OF AN IDEAL FLUID

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In [1] it was found that the vorticity of three-dimensional perturbations in a stationary flow of an ideal fluid increases without bounds in all cases, even if a discrete spectrum of exponentially increasing waves is absent. In general this growth is linear in time and in certain degenerate cases, it even becomes exponential. Below we note that the statement made in [1] does not imply that the three-dimensional perturbations must be unstable, at least in the normally accepted sense. We use an example of three-dimensional perturbations in a plane parallel stationary flow to show, that the velocity of perturbation remains bounded irrespective of the increase in vorticity. We shall briefly recall the arguments used in [1].

Linearizing the equation of vorticity for an ideal homogeneous incompressible fluid relative to the stationary flow $\mathbf{v}_0$, we obtain the following expressions for the perturbations $\mathbf{v}'$:

$$\partial (\mathrm{rot}\, \mathbf{v}') / \partial t + \{\mathbf{v}_0, \mathrm{rot}\, \mathbf{v}'\} + \{\mathbf{v}', \mathrm{rot}\, \mathbf{v}_0\} = 0 \tag{1}$$

$$(\{\mathbf{a}, \mathbf{b}\} = (\mathbf{a}\nabla)\, \mathbf{b} - (\mathbf{b}\nabla)\mathbf{a})$$

The velocity of $\mathbf{v}'$ can be expressed in terms of $\mathrm{rot}\, \mathbf{v}'$ using a fully continuous integral operator. As we know, introduction of such an operator does not affect a continuous spectrum, therefore the continuous spectrum of the problem can be investigated using the following simple equation:

$$\partial (\mathrm{rot}\, \mathbf{v}') / \partial t + \{\mathbf{v}_0, \mathrm{rot}\, \mathbf{v}'\} = 0$$

It is easily verified that the solutions of this equation increase linearly with time.

Let us consider an example. Let the basic flow be directed along the Ox-axis and its velocity be dependent on the z-coordinate, i. e. $U = U(z)$.

For small perturbations harmonically dependent on x and y, i. e. for the perturbations of the form $f(z, t)\ e^{i(\alpha x+\beta y)}$, we have

$$\begin{aligned}
(\partial / \partial t + i\alpha U)\, u + U'w &= - i\alpha p / \rho \\
(\partial / \partial t + i\alpha U)\, v &= - i\beta p / \rho \\
(\partial / \partial t + i\alpha U)\, w &= - p_z / \rho \\
i\alpha u + i\beta v + w_z &= 0
\end{aligned} \tag{2}$$

In this case the equation of vorticity (1) becomes

$$\begin{aligned}
(\partial / \partial t + i\alpha U)\, r_1 - U' r_3 + [- i\beta U' u] &= 0 \\
(\partial / \partial t + i\alpha U)\, r_2 + [- i\beta U' v + U'' w] &= 0 \\
(\partial / \partial t + i\alpha U)\, r_3 + [- i\beta U' w] &= 0
\end{aligned} \tag{3}$$

where r_1, r_2 and r_3 are the components of $\mathrm{rot}\, \mathbf{v}'$,

$$r_1 = i\beta w - v_z,\ r_2 = - i\alpha w + u_z,\ r_3 = i\alpha v - i\beta u \tag{4}$$

The square brackets contain expressions corresponding to the last term of (1). If we discard these terms, then the remaining system of equations relative to r_1, r_2 and r_3 can be integrated without difficulty. Here r_2 and r_3 remain bounded with respect to time, while r_1 increases linearly.

Next we consider the complete system (3) and show, that u, v and w are bounded. Multiplying the first equation by $i\beta$, the second one by $i\alpha$, subtracting the second from the first and taking into account

$$i\beta r_1 - i\alpha r_2 = w_{zz} - (\alpha^2 + \beta^2) w$$

we obtain

$$(\partial / \partial t + i\alpha U) [w_{zz} - (\alpha^2 + \beta^2) w] - i\alpha U''w = 0 \qquad (5)$$

If we have solid walls at $z = a$ and b, then we set at these points $w = 0$.

An imaginary discrete spectrum leading to the exponential instability is absent when e.g. $U'' \neq 0$ (Rayleigh theorem). In this case the following conservation law can be verified without difficulty

$$\frac{d}{dt} \int_a^b \left[|w_z|^2 + (\alpha^2 + \beta^2) |w|^2 + \frac{U + U_0}{U''} |w_{zz} - (\alpha^2 + \beta^2) w|^2 \right] dz = 0$$

This gives the principal part of the functional constructed in [2], while U_0 is an arbitrary constant which can be chosen so that $(U + U_0) / U'' > 0$. The above conservation law implies that w, w_z and w_{zz} are bounded in the quadratic mean by a time-independent constant.

From the third equation of (3) we find

$$r_3 = e^{-i\alpha Ut} \int e^{i\alpha Ut} i\beta U' w dt$$

which, on substitution of w from (5) becomes

$$r_3 = e^{-i\alpha U(z)t} \frac{\beta U'}{\alpha U''} \int e^{i\alpha U t} \left(\frac{\partial}{\partial t} + i\alpha U \right) [w_{zz} - (\alpha^2 + \beta^2) w] \, dt =$$
$$\frac{\beta U'}{\alpha U''} [w_{zz} - (\alpha^2 + \beta^2) w] + e^{-i\alpha Ut} f(z)$$

By virtue of the previous proof this expression is bounded in the quadratic mean by a time-independent constant. Taking into account the last equations of (2) and of (4) we find, that u and v are linearly expressed in terms of r_3 and w_z, i.e. they are also bounded by a time-independent constant, Q.E.D.

Nevertheless, the vorticity components r_1 and r_2 may increase in a linear manner. This can be shown without using any complicated theories by considering the particular solution with $w \equiv 0$. Then r_3 has the form $e^{-i\alpha Ut} f(z)$, consequently, v also has the same form. But $r_1 = -v_z = i\alpha U' tv +, \ldots,$ i.e. it increases linearly with time.

As was already noted in [3], real instability linear with respect to time with increasing velocities, may occur when the points of the disctrete spectrum merge with each other.

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